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EXACT SOLUTION FOR A HIGH-TEMPERATURE JET

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High-temperature gas (plasma) jets are widely used in modern technology and the jet is often laminar (see, e.g., [1]). The Dorodnitsyn transformation used in the study of nonisothermal jets [2], is useful for plane flows with certain limitations placed on the thermophysical properties of the gas and, besides, it is difficult to convert the Dorodnitsyn variables to physical coordinates. An exact similarity solution within the framework of boundary-layer approximations is given in this paper for the nonisothermal axisymmetric flow in the region where the temperature at the jet axis is appreciably higher than the temperature at infinity.

The problem describing the efflux of a nonisothermal jet from a cylindrical orifice can be written within the framework of boundary-layer approximations in the form

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial w}{\partial r}=\rho\left(v\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right),\quad \frac{1}{r}\frac{\partial}{\partial r}r\rho v+\frac{\partial}{\partial z}\rho w=0,\quad \rho T=1,\ \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r}=\Pr\rho\left(v\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right); \tag{1}$$

$$v=\frac{\partial w}{\partial r}=\frac{\partial T}{\partial r}=0 \quad \text{at} \quad r=0 \tag{2}$$

$$T=\varepsilon,\quad w=0 \quad \text{as} \quad r\to\infty. \tag{3}$$

Here r, zRe are cylindrical coordinates (r, z are the inner coordinates in the asymptotic expansion in terms of the small parameter Re^{-1}); $\mathrm{Re}=\sqrt{\rho_M I_{1M}/2\pi}/\mu_M$ is a certain analogous Reynolds number; $v\mathrm{Re}^{-1}$, w are the longitudinal and transverse velocity components; $\Pr=c_{pM}\mu_M/\lambda_M$ is the Prandtl number; ε is the value of the temperature at infinity; the rest are conventional quantities. In order to nondimensionalize, the quantities T_M, ρ_M, c_{pM}, μ_M, and λ_M (dimensional quantities are denoted by the subscript M), and also the values of total impulse I_{1M} and flow enthalpy I_{2M} given by the equations

$$I_{1M}=2\pi\rho_M V_M^2 L_M^2\int_0^\infty \rho w^2 r\,dr,\quad I_{2M}=2\pi c_{pM}\rho_M T_M V_M L_M^2\int_0^\infty \rho w(T-\varepsilon)\,r\,dr$$

are assumed specified. The reference scales for the velocity V_M and the length L_M are given by

$$V_M=c_{pM}T_M I_{1M}/I_{2M},\ L_M=I_{2M}/(c_{pM}T_M\sqrt{2\pi\rho_M I_{1M}}).$$

In writing Eqs. (1) it was assumed that the specific heat, thermal conductivity, and dynamic viscosity are constants. For the problem (1)-(3) the initial conditions should have been fixed at $z=z_0$ but in the present study only similarity solutions will be considered and hence in order to complete the set of equations for the problem (1)-(3), we formulate conditions for the conservation of momentum and enthalpy

$$\int_0^\infty \rho w^2 r\,dr=1,\quad \int_0^\infty \rho w(T-\varepsilon)\,r\,dr=1. \tag{4}$$

The problem (1)-(4) will be considered as $\varepsilon\to 0$. In the zeroth-order approximation in terms of ε, the problem (1)-(4) is transformed to the system of equations (1), boundary conditions (2), and

$$w=T=0 \quad \text{as} \quad r\to\infty; \tag{5}$$

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the integral relations (4) in this case have the form

$$\int_0^\infty \rho w^2 r dr = 1, \quad \int_0^\infty w r dr = 1. \tag{6}$$

The similarity solution for the problem (1), (2), (5), and (6) is sought in the form

$$w(r, z) = z^{\alpha_w} u(x), \; v(r, z) = z^{\alpha_v} f(x), \; T(r, z) = z^{\alpha_T} \theta(x), \; r = z^{\alpha} x, \tag{7}$$

where the similarity parameters $\alpha_w, \ldots, \alpha$ are determined from Eq. (1), integral relations (6), and are given by

$$\alpha_w = \alpha_T = -1, \; \alpha = 1/2, \; \alpha_v = -3/2. \tag{8}$$

Substituting (7), (8) in Eqs. (1), (2), (5), and (6), it is possible to obtain

$$(1/x)(xu')' = (1/\theta)[fu' - u(u + (1/2)xu')], \; (1/x)(xf/\theta)' - (1/2)(u/\theta)' = 0, \; (1/x)(x\theta')' = \Pr(1/\theta)[f\theta' - u(\theta + (1/2)x\theta')]; \tag{9}$$

$$f = u' = \theta' = 0 \quad \text{at} \quad x = 0; \tag{10}$$

$$\theta = u = 0 \quad \text{as} \quad x \to \infty; \tag{11}$$

$$\int_0^\infty \frac{u^2}{\theta} x dx = 1, \quad \int_0^\infty u x dx = 1. \tag{12}$$

Note that the solutions to the problem (9)-(11) are invariant to the transformation

$$x \to ax, \; \theta \to b\theta, \; u \to a^{-2}bu, \; f \to a^{-1}bf, \tag{13}$$

where a, b are arbitrary constants, i.e., the boundary-value problem (9)-(11) can be transformed to Cauchy problem by arbitrarily giving the necessary initial conditions at x = 0, e.g.,

$$\theta = u = 1 \quad \text{at} \quad x = 0. \tag{14}$$

Then, after integrating the system (9) with initial conditions (10) and (14), it is necessary to know whether or not the solution obtained for the Cauchy problem satisfies the boundary conditions at infinity (11). If the solution to the Cauchy problem satisfies these conditions, then it is the nontrivial solution to the problem (9)-(11), which can be transformed to the form satisfying the integral conditions (12) with the help of the invariant properties (13). The system of equations (9) is transformed to the form

$$(1/x)(xu')' = (1/\theta)(su'/\Pr - u^2), \; (1/x)(x\theta')' = \theta' s/\theta - \Pr u, \; (1/x)(xs)' = \theta' s/\theta - \Pr u, \tag{15}$$

where

$$s = \Pr(f - (1/2)xu). \tag{16}$$

Boundary conditions (10), (11) are then written in the form

$$\theta' = u' = s = 0 \quad \text{at} \quad x = 0, \; \theta = u = 0 \quad \text{as} \quad x \to \infty. \tag{17}$$

It is possible to observe from the problem (16), (17) that the equations and boundary conditions for s and θ' coincide, i.e.,

$$\theta' = s. \tag{18}$$

The solutions for u is sought in the form

$$u = \theta^\beta. \tag{19}$$

Substituting (19) in the first two equations of the system (15) and keeping in view (18), we get

$$\begin{gathered}(1/x)(x\theta')' = (\theta'/\theta)(1/\Pr - \beta + 1) - (1/\beta)\theta^\beta, \\ (1/x)(x\theta')' = \theta'^2/\theta - \Pr\theta^\beta.\end{gathered} \tag{20}$$

The two equations of the system (20) will not be contradictory if we put

$$\beta = 1/\Pr. \tag{21}$$

Then, with the consideration of (18), (19), and (21) the problem (15), (17) is reduced to a simpler, single second-order equation

$$(1/x)(x\theta')' = \theta'^2/\theta - \Pr\theta^{1/\Pr} \tag{22}$$

with boundary conditions

$$\theta'|_{x=0} = 0, \ \theta|_{x\to\infty} = 0. \tag{23}$$

The problem (22), (23) for Pr = 1 allows an extremely simple solution

$$\theta = A_0 x^A \exp(-x^2/4). \tag{24}$$

Assuming that the temperature at the jet axis is finite and nonzero, we put $A=0$, and the constant A_0 is determined from the normalization conditions (12). Then, according to (19), (21) and from Eqs. (24) it is possible to obtain for Pr = 1

$$u = \theta = (1/2)\exp(-x^2/4), \tag{25}$$

and the radial velocity, as follows from (16)-(18), is equal to zero

$$f = \theta' - (1/2)x\theta = 0. \tag{26}$$

Thus, when Pr = 1, the quantity u/θ or (see Eqs. (7), (8)) $\rho w = 1$ remain constant in the entire flow region of the high-temperature jet, i.e., an increase in temperature causes a proportional increase in axial velocity while the radial velocity (26) remains equal to zero.

When Pr ≠ 1 the solution of the problem (22), (23) is considerably more difficult to obtain. We shall now describe the procedure for its solution. Let $\theta = g^{\mathrm{Pr}/1-\mathrm{Pr})}$, then Eq. (22) is transformed to the form

$$(1/x)(xg')' = g'^2/g - (1-\mathrm{Pr})g^2. \tag{27}$$

After the introduction of a new function $g = \exp(y)$ the Eq. (27) is rewritten in the form

$$(1/x)(xy')' = -(1-\mathrm{Pr})\exp(y). \tag{28}$$

The Eq. (28) is a particular form of the Emden–Fowler equation [3]. The procedure for its (Eq. (28)) solution is as follows. By introducing a new function $\eta(t)$ and an argument t

$$\eta(t) = xy', \ t = x^2 \exp(y) \tag{29}$$

it is possible to obtain from Eq. (28)

$$\eta^2 + 4\eta + 2(1-\mathrm{Pr})t + 4c = 0, \tag{30}$$

where c is a constant determined subsequently as the coefficient of g^2 in Eq. (27). Using Eq. (29) with the help of (30), (28) we get the Riccati equation $2x^2y'' - 2xy' - x^2y'^2 - 4c = 0$, which can be reduced to the Euler equation with the help of the transformation $y' = -2q'/q$

$$x^2q'' - xq' + cq = 0. \tag{31}$$

The solution of Eq. (31) is obvious: $q = C_1x^{1+\sqrt{1-c}} + C_2x^{1-\sqrt{1-c}}$. Inverse transformation to the function θ gives

$$\theta = \left(C_1x^{1+\sqrt{(1-\mathrm{Pr})/(8C_1C_2)}} + C_2x^{1-\sqrt{(1-\mathrm{Pr})/(8C_1C_2)}}\right)^{-2\mathrm{Pr}/(1-\mathrm{Pr})}. \tag{32}$$

Assuming that θ is finite and nonzero at $x = 0$, it is necessary to equate to zero the exponent of x in Eq. (32):

$$1 - \sqrt{(1-\mathrm{Pr})/(8C_1C_2)} = 0. \tag{33}$$

C_1 is determined from Eq. (33) and substituted in (32) to get

$$\theta = (C_2 + ((1-\mathrm{Pr})/8C_2)x^2)^{-2\mathrm{Pr}/(\mathrm{Pr}-1)}. \tag{34}$$

It follows from Eq. (34) that when Pr > 1 there is no solution that decreases as $x \to \infty$. This can be explained by the following possible reasons: 1) the similarity solution (34) is applicable within a limited range of the variable x (i.e., in the problem (1), (2), (5), and (6) the boundary conditions $w = T = 0$ are not set at $r \to \infty$, but at a finite value of r, or, in other words, the boundary-layer thickness is finite; 2) there is no similarity solution for the problems (1), (2), (5), and (6) in the form (7), (8).

Using Eqs. (34), (19), (21), (18), and (16), invariant properties (13), and integral relations (12), we obtain solutions for θ, u, and f for Pr < 1 in the form

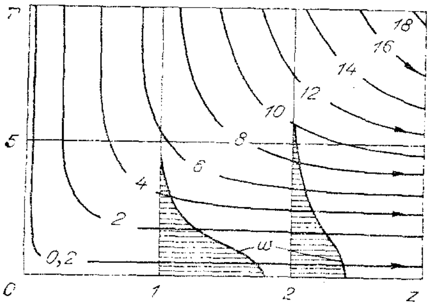

Fig. 1

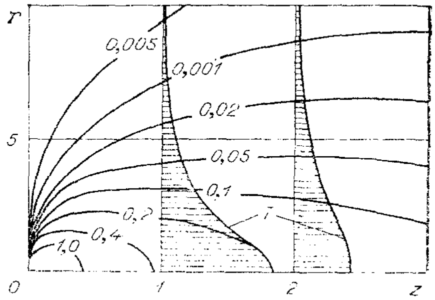

Fig. 2

$$\theta = \frac{1+\mathrm{Pr}}{4}\left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}x^2\right]^{-2\mathrm{Pr}/(1-\mathrm{Pr})}, \tag{35}$$

$$u = \frac{3-\mathrm{Pr}}{4}\left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}x^2\right]^{-2/(1-\mathrm{Pr})},$$

$$f = \frac{3-\mathrm{Pr}}{8}x\left\{\left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}x^2\right]^{-2/(1-\mathrm{Pr})} - \left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}x^2\right]^{-\frac{1+\mathrm{Pr}}{1-\mathrm{Pr}}}\right\}.$$

Using Eqs. (7) and (8) we revert to variabales r, z and functions w, T, and v with the help of Eq. (35):

$$T = \frac{1+\mathrm{Pr}}{4}\,\frac{1}{z}\left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}\,\frac{r^2}{z}\right]^{-2\mathrm{Pr}/(1-\mathrm{Pr})}, \tag{36}$$

$$w = \frac{3-\mathrm{Pr}}{4}\,\frac{1}{z}\left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}\,\frac{r^2}{z}\right]^{-2/(1-\mathrm{Pr})},$$

$$v = \frac{3-\mathrm{Pr}}{8}\,\frac{r}{z^2}\left\{\left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}\,\frac{r^2}{z}\right]^{-2/(1-\mathrm{Pr})} - \left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}\,\frac{r^2}{z}\right]^{-\frac{1+\mathrm{Pr}}{1-\mathrm{Pr}}}\right\}.$$

Introducing the stream function

$$\psi = \int_0^r r\rho w\,dr \tag{37}$$

and using Eq. (36), we get

$$\psi = \frac{4}{1-\mathrm{Pr}}\,z\left\{1 - \left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}\,\frac{r^2}{z}\right]^{-1}\right\}. \tag{38}$$

When Pr → 1, the stream function (37) does not depend on z:

$$\lim_{\mathrm{Pr}\to 1}\psi = \lim_{\mathrm{Pr}\to 1}\frac{4z}{1-\mathrm{Pr}}\left\{1 - \left[1 + \frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(1+\mathrm{Pr})}\,\frac{r^2}{z}\right]^{-1}\right\} = \frac{r^2}{2}. \tag{39}$$

As r tends to infinity in the equation for ψ (38), we get

$$\psi|_{r\to\infty} = 4/(1-\mathrm{Pr}), \tag{40}$$

where $\psi|_{r\to\infty}$ can be interpreted as a mass of fluid in the jet per unit time, or the total discharge, determined accurately to the coefficient 2π. It follows from Eq. (40) that the total discharge, as in the case of incompressible fluid, increases with distance from the nozzle. When Pr → 1, the total discharge increases unboundedly, and Eq. (40) becomes inapplicable at Pr = 1. In this case, it follows from Eq. (39) that the stream function does not depend on z and the total discharge becomes infinite.

Streamlines shown in Fig. 1 for a laminar high-temperature circular jet for Pr = 1/2 are obtained from the equation

$$r^2 = \frac{8(1+\mathrm{Pr})}{(3-\mathrm{Pr})(1-\mathrm{Pr})}\,z\left[\left(1 - \frac{1-\mathrm{Pr}}{4}\,\frac{\psi}{z}\right)^{-1} - 1\right], \tag{41}$$

where Pr and ψ are considered known constants. As $z \to \infty$, it follows from Eq. (41), $r^2|_{r \to \infty} = (2(1+Pr)/(3-Pr))\psi$, i.e., the streamlines become parallel to the z axis. At $z = z_p = ((1-Pr)/4)\psi$, the function $r^2 = r^2(z)$ has a pole, i.e., cold gas is entrained from infinity (radially) along the surface $z = z_p$ and heated. Isotherms determined from the first of Eqs. (36) are shown in Fig. 2 for $Pr = 1/2$.

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